FINAL REPORT: TOPOLOGICAL METHODS IN AUTOMORPHIC FORMS

MICHAEL J. HOPKINS

Abstract

The work supported by this grant resulted in the solution by Michael Hill, Michael Hopkins and Doug Ravenel of the "Kervaire Invariant" problem. The Kervaire invariant problem was one of the longest standing open problems in algebraic topology, and its solution represents major breakthrough. The many new innovative techniques developed for this purpose are making a strong impact on the field of algebraic topology. The solution hints at a relationship between quantum field theory in dimension four and the Kervaire invariant, making an unexpected connection between this old problem and theoretical physics.

Scientific findings

Topological automorphic forms. The initial phase of this program involved studying the interaction between the theory of topological automorphic forms, and the "Hopkins-Miller" cohomology theories EO_n. The hope was to use the theory of Shimura varieties to prove an old conjecture of Hopkins on the action of finite subgroups of the Morava stabilizer group on the homotopy groups of E_n. The first question one faces is to relate the theories at all: given a prime p, and a chromatic level n, does there exist a unitary Shimura variety such that there is an equivalence

$$EO_n \simeq \mathrm{TAF}_{K(n)}?$$

Behrens and Lawson [1] have shown that there is an equivalence

$$\mathrm{TAF}_{K(n)} \simeq \left(\prod_i E_n^{hG_i}\right)^{h(p)}.$$

Here, the number of terms in the product corresponds to a certain non-abelian class number for the unitary group U, and the groups G_i are finite subgroups of the Morava stabilizer group. The groups G_i are the automorphism groups of mod p points of the associated Shimura stack. The problem is therefore equivalent to finding Shimura data for which there exists an i such that G_i is a maximal finite subgroup with maximal p-torsion.

Behrens and Hopkins were able to determine that for $n = (p-1)p^{r-1}$, there is a *canonical* choice of Shimura variety having the desired mod p-point, answering this question in the affirmative. The result appears in [2].

At this point two unexpected developments occurred. First, Hill, Hopkins and Ravenel, using new methods, proved the conjecture of Hopkins on the action of finite subgroups of the Morava stabilizer group. Their work did not make use of the theory of Shimura varieties, and raised the possibility of reversing the intended

relationship and using these methods to make new computations in the theory of topological automorphic forms. It also allowed Hill, Hopkins and Ravenel to make many new computations in chromatic homotopy theory. This led to the second unexpected development. Using the information from these new computations, Hill, Hopkins and Ravenel solved the longstanding "Kervaire invariant problem."

The Kervaire invariant problem. The Kervaire invariant problem is one of the oldest problems in differential and algebraic topology. It originates in the 1930's in the work of Pontryagin describing the maps between spheres in terms of the geometry of the inverse image of a regular value. Pontryagin introduced the rudiments of what is now know as the theory of "surgery" on manifolds, and offered an incorrect argument that every framed manifold of dimension 2 could be transformed using surgery into the sphere. He later discovered his error, and the somewhat subtle invariant he had overlooked.

The first form of the Kervaire invariant problem has its origins in Pontryagin's work of the 1930's.

Question 1. When can a stably framed manifold be transformed into a sphere. More precisely, when is a framed manifold cobordant to a homotopy sphere.

The topic was taken up again in late 1950's and early 1960's by Thom, Milnor, Kervaire and others. Milnor startled the mathematical world in 1957 by constructing a smooth manifold homeomorphic, but not diffeomorphic to the sphere S^7 of dimension 7. He also introduced the theory of "surgery" and made great strides in classifying manifolds using surgery to transform one into another. Kervaire discovered the deeper nature of Pontryagin's subtle error and defined an important invariant, the *Kervaire invariant* for (almost) framed manifolds of dimension $(4k+2)$. Using it he produced an example of a triangulable manifold of dimension 10 which did not admit a smooth structure. Working together, Kervaire and Milnor introduced the group Θ_n of smooth structures on (homotopy) spheres of dimension n, with the group operation of connected sum. For n congruent to 0 or 3 mod 4 they were able to determine Θ_n in terms of the homotopy groups of spheres. For n congruent to 1 or 2 mod 4 they were unable to settle a factor of 2 in the order of Θ_n. They showed that settling the mysterious factor of 2 was equivalent to the question stated above. They also showed that the question is equivalent to

Question 2. In which dimension n can there exist a smooth stably framed manifold with Kervaire invariant 1.

By the mid 1960's little progress had been made on this question, and the available information was largely anectodal. It was known that manifolds of Kervaire invariant one existed in dimensions 2, 6 and 14, and Kervaire had shown that his invariant was always zero in dimensions 10 and 18. The difficulty was the lack of a robust homotopy theoretic description of Kervaire's invariant. The next significant step was taken in the 1966 paper of Brown and Peterson who showed that no framed manifold of Kervaire invariant one could exist in dimension of the form $(8k+2)$ with $k \geq 1$, thus extending Kervaire's sequence of 10 and 18. But the definitive result came a couple of years later, in the famous 1969 paper of Browder, "The Kervaire invariant of framed manifolds and its generalizations." Browder showed that if a framed manifold of Kervaire invariant 1 were to exist, the dimension of the manifold must be of the form $2^{j+1}-2$, and in that case that such a manifold existed

if an only if there was an element θ_j in the homotopy groups of spheres represented at the E_2-term of the classical Adams spectral sequence by h_j^2. With Browder's work, the methods of homotopy theory could be brought to bear on the problem, and Barratt and Mahowald and their co-workers Tangora and Jones extended the sequence of existence results to include 30 and 62. The question was also related to many deep issues in homotopy theory, especially in connection with the homotopy groups of spheres. By the mid 1980's Kervaire invariant one manifolds were known to exist in dimensions 2, 6, 14, 30, and 62, and the issue was completely open in dimensions $2^{j+1}-2$ for $j \geq 6$. No significant progress was made after that.

Under the support of this grant, Mike Hill, Hopkins and Doug Ravenel proved the following result.

Theorem 3. *If M is a stably framed smooth, closed manifold of Kervaire invariant one, the the dimension of M is* 2, 6, 14, 30, 62, *or* 126.

This settles this problem in all dimensions except 126.

The solution to the Kervaire invariant problem involves two key technical innovations. One is the introduction of the $\mathbb{Z}/2^{n+1}$-equivariant E_∞ ring spectrum $MU^{((n))}$. The spectrum $MU^{((0))}$ is Landweber's spectrum $MU_{\mathbb{R}}$ of *real bordism* equipped with the action of the Galois group of $\mathbb{C}$ over $\mathbb{R}$ (which is cyclic of order 2). The spectrum $MU^{((n))}$ is the 2^n-fold smash product of $MU_{\mathbb{R}}$ with the action of $\mathbb{Z}/2^n$ generated by

$$a_1 \wedge \cdots \wedge a_{2^n} \mapsto \bar{a}_{2^n} \wedge a_1 \wedge \cdots \wedge a_{2^n-1}.$$

The other is an equivariant refinement of the Postnikov tower called the *slice filtration*. The slice filtration is engineered to produce a convenient output when the group in question is cyclic of order a power of 2, especially in the case of $MU^{((n))}$. These new tools harness a remarkable range of information in topology.

Activities and supported personnel

Conferences

Summer Topology Program June-August, 2008
Summer Topology Program June-August, 2009
Workshop on Hirzebruch Genera Associated to Algebraic Curves, July 2010

Personnel

Faculty	Michael Hopkins (Principal Investigator) Dan Dugger (University of Oregon) Mike Hill (University of Virginia) Mark Mahowald (Northwestern University) David Nadler (Northwestern University) Doug Ravenel (University of Rochester) Isadore Singer (MIT) Jeffrey Smith (University of British Columbia) Markus Szymik (Ruhr-Universitt Bochum) Paul Windey (Jussieu)
Post Doctoral Fellow	Mark Behrens
Graduate students	Samik Basu, Ph.D awarded 2009 Reid Barton, Current Graduate Student Sam Isaakson, Ph.D awarded 2009

Dissemination

Invited Addresses

M. J. Hopkins

Applications of algebra to a problem in topology, Atiyah80, Edinburgh, April 2009

The Kervaire Invariant problem, Perspectives in Mathematics and Physics: A Conference in Honor of I.M. Singer's 85th birthday, May 2009

Takagi Lectures, Tokyo, November 2009

On the Kervaire Invariant problem, Current Events Bulletin, winter AMS meeting, January 2010

On the Kervaire Invariant problem, Inauguration: Centre for Symmetry and Deformation, Jan, 2010, Copenhagen

Colloquium lecture, Princeton University, Feb. 2010

Pinsky Lecture Series, Northwestern University, Feb. 2010

Mathematical invariants: how to know the answer in advance, Public Dialogue, Aspen center for physics, July 2010

Invited Addresses (ctd)

D. C. Ravenel	Instituto Superior Tcnico, Lisbon, May, 2009
	AMS special session on homotopy theory Penn State, October, 2009
	Tokyo Metropolitan University, October, 2009
	Current Developments in Math conference at Harvard, November, 2009
	Princeton University Workshop, February 11, 2010
	University of Western Ontario, April, 2010
	Cornell Topology festival, May, 2010
	Bilkent and Middle East Technical Universities, Ankara, May, 2010
	Gokova Geometry and Topology Conference, Turkey, June, 2010
M. A. Hill	Oberwolfach Lecture Series, September 2010: *The Kervaire Invariant One Problem*
	Homotopy Theory and Derived Algebraic Geometry, September 2010: *Equivariant Computations and the Kervaire Invariant*
	Cascades Topology Conference, Banff, April 2010: *Equivariant homotopy around the Kervaire Invariant One problem*
	Informal Workshop on the solution by Hill, Hopkins, and Ravenel of the Kervaire Invariant Problem, Princeton University, February 2010: *Equivariant Computations and the Gap Theorem*
	Indiana University Colloquium, December 2010: *On the Non-Existence of Kervaire Invariant One Manifolds*
	Current Developments in Mathematics Conference, Harvard University, November 2009: *The Arf-Kervaire Problem in Algebraic Topology*
	Northwestern University Colloquium, October 2009
	University of Illinois at Urbana-Champaign Lecture Series, September 2009
	University of Illinois at Urbana-Champaign Colloquium, September 2009: *On the Non-Existence of Kervaire Invariant One Manifolds*
	University of Oslo, August 2009: *The Slice Spectral Sequence*
	Conference on p-adic Geometry, Norway, August 2009: *On the slice filtration and the Kervaire invariant one problem*
	Isle of Skye Algebra and Topology Conference, June 2009: *On the Non-Existence of Kervaire Invariant One Manifolds*
	Georgia Topology Conference, May 2009: *On the Non-Existence of Manifolds of Kervaire Invariant One*

Publications.

[1] Mark Behrens and Tyler Lawson. Topological automorphic forms. *Mem. Amer. Math. Soc.*, 204(958):xxiv+141, 2010.

[2] Mark Behrens and Michael J. Hopkins. Higher real K-theories and topological automorphic forms. *J. Topology*, page 37, 2011.

[3] Markus Szymik. $K3$ spectra. *Bull. Lond. Math. Soc.*, 42(1):137–148, 2010.

[4] Daniel Dugger and Daniel C. Isaksen. The motivic Adams spectral sequence. *Geom. Topol.*, 14(2):967–1014, 2010.

[5] Mark Behrens and Daniel G. Davis. The homotopy fixed point spectra of profinite Galois extensions. *Trans. Amer. Math. Soc.*, 362(9):4983–5042, 2010.

[6] Mark Behrens and Gerd Laures. β-family congruences and the f-invariant. In *New topological contexts for Galois theory and algebraic geometry (BIRS 2008)*, volume 16 of *Geom. Topol. Monogr.*, pages 9–29. Geom. Topol. Publ., Coventry, 2009.

[7] Mark Behrens. Congruences between modular forms given by the divided β family in homotopy theory. *Geom. Topol.*, 13(1):319–357, 2009.

[8] Mark Behrens and Tyler Lawson. Topological automorphic forms on $u(1,1)$. *Math. Zeit*, 2011.

[9] Daniel S. Freed, Michael J. Hopkins, and Constantin Teleman. Consistent orientation of moduli spaces. In *The many facets of geometry*, pages 395–419. Oxford Univ. Press, Oxford, 2010.

[10] Daniel S. Freed, Michael J. Hopkins, Jacob Lurie, and Constantin Teleman. Topological quantum field theories from compact Lie groups. In *A celebration of the mathematical legacy of Raoul Bott*, volume 50 of *CRM Proc. Lecture Notes*, pages 367–403. Amer. Math. Soc., Providence, RI, 2010.

[11] Michael A. Hill, Michael J. Hopkins, and Douglas C. Ravenel. On the non-existence of elements of kervaire invariant one. arXiv:0908.3724.

[12] Michael A. Hill, Michael J. Hopkins, and Douglas C. Ravenel. The Arf-Kervaire problem in algebraic topology: sketch of the proof. Current Developments in Mathematics, 2009.

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